Exact Downscaling in Geostatistical Modeling

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Issues of scale pervade geostatistics. We deal with data of different scale and are often required to upscale and downscale models. This research is aimed at constructing scale-consistent models at different scales. Coarse scale models are often constructed over large areas consistent with larger scale seismic data and geologic trends. Then, finer scale models are constructed to represent the heterogeneity at a scale that affects recovery calculations. A number of models at different scale may be constructed. Fine scale models must be constructed so that they average exactly to the large scale models. This is required to avoid conflicts between different models and to capture the trends and large scale seismic data contained in the large scale models.

The theory of exact reproduction of large scale data in kriging and simulation is developed. A number of numerical and analytical examples are presented. A number of theoretical, implementation and practical aspects are illustrated.

Introduction

Downscaling has often been used to deal with multiscale data in geostatistical modeling. Large scale data is commonly available for geostatistical modeling. The large scale data (block data) may come from: (1) a 2-D large scale prior model, which usually is built over a very large area; (2) inversion from production data and well testing data, which usually have a much larger scale than the core and well log data; (3) inversion from seismic data; (4) geologic trend data that is considered reliable; (5) any other large scale data from measurement or expert interpretation. Downscaling is required when building a model from these large scale data. Downscaling is also used to generate fine scale 3D models from large scale models for detailed flow simulation on small areas, such as pad area. The large scale model could be a 3D coarser grid model or a 2D model where the downscaling is actually extending the 2D model to 3D model.

Some of the techniques used for downscaling with multiscale data include cokriging, collocated cokriging, and using the block data as locally varying mean data. None of these methods reproduce the block data exactly. To avoid any inconsistencies between models and between models and data, it is desired that the fine scale model average up to the correct large scale model. Of course, if the large scale data is a different measurement or contains errors, then the reproduction of the block data has a different meaning. This paper focuses on the exact reproduction of block data in scaling.

To keep consistency between the original model and downscaled model, a simple duplication method could be used; the block values are replicated for each constituent small cell in the block. Clearly, the heterogeneity likely to exist at the fine scale will not be preserved. Our goal is to generate a fine scale model with correct fine scale spatial correlation *and* exact reproduction of

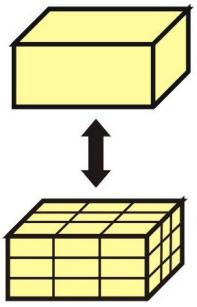
the block data at the same time. Block kriging and direct sequential simulation are combined for downscaling. The background information about the scaling is presented first. Then, the theory of the downscaling techniques is presented. An analytical demonstration of the exact reproduction of block data is provided in the Appendix. A numerical demonstration with different block data, well data, and variogram models are given after the theory. A large example of extending a 2D porosity model to 3D model is provided. At the end, some considerations of the downscaling technique are discussed.

Background

Consider a block with an arbitrary shape. The block is composed of *n* small grid cells. If there are values or point data in all of small cells, the block value can be calculated from the point data by this equation:

$$Z_V(\mathbf{u}) = \frac{1}{V}\int Z(\mathbf{u}')d\,\mathbf{u}' \tag{1}$$

where $Z(\mathbf{u}')$ is the point data, and $Z_V(\mathbf{u})$ is the block datum. If the sizes of the n small cells are equal, the equation becomes a simple linear equation:



$$Z_V(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n} Z(\mathbf{u}_i) \tag{2}$$

where $Z(\mathbf{u}_i)$ $(i = 1, 2, \ldots n)$ is the point data, and $Z_V(\mathbf{u})$ is the block datum. This equation is linear. Some petrophysical properties, such as porosity and water saturation, have the linear relationship between data at different scales. Recall that the porosity is the fraction of pore volume V_{pV} over the bulk volume of the rock V_{bV}. Discretizing the bulk volume into *n* equal sized small bulk volumes V_{bv}, the following equation (3) shows that the block porosity value $\phi_V(\mathbf{u})$ is exact the average of the point porosities $\phi(\mathbf{u}_i)$:

$$\phi_V(\mathbf{u}) = \frac{V_{pV}}{V_{bV}} = \frac{\sum_{i=1}^{n} V_{pi}}{n\,V_{bv}} = \frac{1}{n}\sum_{i=1}^{n}\frac{V_{pi}}{V_{bv}} = \frac{1}{n}\sum_{i=1}^{n}\phi(\mathbf{u}_i) \tag{3}$$

It is same for water saturation. Block averaging is exact. Permeability does not average linearly; however, a power law transformation can be used to make the arithmetic averaging applicable to permeability (Deutsch, 1987 and CCG Report 2002).

Downscaling process is not as straightforward as upscaling. In downscaling, the heterogeneity at smaller scales must be taken into account. The block data cannot uniquely provide the heterogeneity at the fine scale. The pattern of heterogeneity comes through some statistical parameters such as variograms/covariances or training images. The uncertainty comes through multiple realizations that quantify the uncertainty. We focus our attention on variogram-based geostatistics in this paper. The point scale variogram model can be generated from well data. The non-linear relationship in heterogeneity at different scales can be accounted for by calculating volume averaged variogram or covariance values:

$$C_V(\mathbf{h}) = E\{Z_V(\mathbf{u}) \bullet Z_V(\mathbf{u}+\mathbf{h})\} - E\{Z_V(\mathbf{u})\}E\{Z_V(\mathbf{u}+\mathbf{h})\} = E\{Z_V(\mathbf{u}) \bullet Z_V(\mathbf{u}+\mathbf{h})\} - m^2$$

$$= E\left\{\frac{1}{|V|}\int Z(\mathbf{u}')d\,\mathbf{u}' \bullet \frac{1}{|V|}\int Z(\mathbf{u}''+\mathbf{h})d\,\mathbf{u}''\right\} - m^2$$

$$= E\left\{\frac{1}{|V||V|}\iint Z(\mathbf{u}')Z(\mathbf{u}''+\mathbf{h})d\,\mathbf{u}'\,d\,\mathbf{u}''\right\} - m^2$$

$$= \frac{1}{|V||V|}\iint (E\{Z(\mathbf{u}')Z(\mathbf{u}''+\mathbf{h})\} - m^2)d\,\mathbf{u}'\,d\,\mathbf{u}''$$

$$= \frac{1}{V^2}\iint C(\mathbf{h})d\,\mathbf{u}'\,d\,\mathbf{u}''$$

$$C\{Z_V(\mathbf{u}), Z(\mathbf{u}+\mathbf{h})\} = E\{Z_V(\mathbf{u}) \cdot Z(\mathbf{u}+\mathbf{h})\} - E\{Z_V(\mathbf{u})\}E\{Z(\mathbf{u}+\mathbf{h})\}$$

$$= E\{Z_V(\mathbf{u}) \cdot Z(\mathbf{u}+\mathbf{h})\} - m^2 = E\left\{\frac{1}{|V|}\int Z(\mathbf{u}')d\,\mathbf{u}' \cdot Z(\mathbf{u}+\mathbf{h})\right\} - m^2$$

$$= \frac{1}{|V|}\int E\{Z(\mathbf{u}')d\,\mathbf{u}' \cdot Z(\mathbf{u}+\mathbf{h})\} - m^2$$

$$= \frac{1}{|V|}\int E\{Z(\mathbf{u}') \cdot Z(\mathbf{u}+\mathbf{h})\} - m^2)d\,\mathbf{u}'$$

$$= \frac{1}{|V|}\int C(\mathbf{h})d\,\mathbf{u}'$$

In downscaling of the block data to a fine scale model, the fine scale model is conditioned to available well data. Downscaling is a multiscale modeling problem using block data and point data. A direct sequential simulation framework can be used for the downscaling.

Theory

Consider a random function Z(**u**) distributed over a field *D*: $\{Z(\mathbf{u_i}), \forall\, \mathbf{u_i} \in D\}$. Assume second-order stationarity, so *m,* σ and *C*(**h**) are constant over *D*. Given the block data and additional point data, each small cell can be estimated using a multiscale simple kriging technique:

$$Z^*(\mathbf{u}_i) = \sum_{k=1}^{K} \lambda\, Z_V(\mathbf{u}_k) + \sum_{j=1}^{m} w_j Z(\mathbf{u}_j) \tag{4}$$

where $Z^*(\mathbf{u}_l)$ is the kriging estimate at the fine grid cell $\mathbf{u}_i$, $Z_V(\mathbf{u}_k)$ is block data, and $Z(\mathbf{u}_j)$ is point data. Similarly, the block data and the point data can be used to simulate a value at each small cell using a multiscale direct sequential simulation technique:

$$Z_s(\mathbf{u}_i) = \sum_{k=1}^{K} \lambda\, Z_V(\mathbf{u}_k) + \sum_{j=1}^{m} \mu_j Z(\mathbf{u}_j) + \sum_{l=1}^{i-1} w_l Z_s(\mathbf{u}_l) + R(\mathbf{u}_i) \tag{5}$$

Where $Z_s(\mathbf{u}_i)$ is the simulated value at the fine grid cell $\mathbf{u}_i$, and $Z(\mathbf{u}_j)$ is the point data, $Z_s(\mathbf{u}_l)$ is the previously simulated datum, $R(\mathbf{u}_i)$ is the random residual at the location $\mathbf{u}_i$.

According to the properties of kriging and simulation techniques, the downscaling will generate a fine scale model that follows the correct spatial correlation and reproduces the point data. In the downscaling, the block is also expected to be reproduced so that the fine scale model is consistent with the original model and no bias is introduced by the scaling process. The interesting thing is that, after the downscaling by these two methods, the block average of the estimated or simulated values is exactly the block datum used in the downscaling. Two theorems on the exact reproduction of block data using multiscale kriging and direct sequential simulation techniques are given below. The proofs of these are given later.

Theorem I (EDMK): kriging with block and point data is an exact downscaling method:

$$Z^*(\mathbf{u}_\mathbf{i}) = \sum_{k=1}^{K} \lambda Z_V(\mathbf{u}_k) + \sum_{j=1}^{m} w_j Z(\mathbf{u}_j)$$

That is, the estimated values of the small cells in a block can exactly reproduce the block values:

$$Z_V^*(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n} Z^*(\mathbf{u}_\mathbf{i}) = Z_V(\mathbf{u})$$

Theorem II (EDMDSS): simulation using a direct sequential simulation is an exact downscaling method:

$$Z_s(\mathbf{u}_i) = \sum_{k=1}^{K} \lambda Z_V(\mathbf{u}_k) + \sum_{j=1}^{m} \mu_j Z(\mathbf{u}_j) + \sum_{l=1}^{i-1} w_l Z_s(\mathbf{u}_l) + R(\mathbf{u}_i)$$

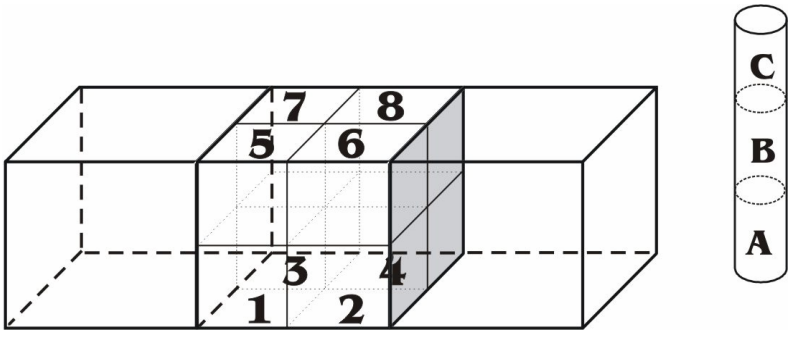
That is, the simulated values of the small cells in a block can exactly reproduce the block value:

$$Z_{Vs}(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n} Z_s(\mathbf{u}_\mathbf{i}) = Z_V(\mathbf{u})$$

Theorem I is *Exact Downscaling with Multiscale Kriging* (EDMK) and Theorem II is *Exact Downscaling with Multiscale Direct Sequential Simulation* (EDMDSS). The exact reproduction of the block data in the two theorems are analytically demonstrated in the Appendix. A numerical illustration of the block datum reproduction using EDMK is shown next.

Illustration with small examples:

Consider three block data in a row and three point data in a vertical well close to the blocks, the center block is going to be downscaled into 8 small cells as shown in the figure below.



The data are listed in the table below. The center block datum is changed in all three sets. The two adjacent block data change in sets 1 and 2, while the well data are kept same for checking the effect of block data. Then, the well data change in sets 2 and 3, while the two adjacent block data are kept same for checking the effect of well data.

	Location	Set 1	Set 2	Set 3
Block	Left	0.50	0.80	0.80
	Center	0.25	1.00	4.00
	Right	0.10	1.20	1.20
Well	A	0.70	0.70	0.50
	B	0.50	0.50	1.20
	C	2.00	2.00	3.00

An exponential variogram model is used for the downscaling: $\gamma(h) = 1 - e^{-\frac{h}{a}}$. To check the block datum reproduction under different variograms, three different variogram ranges are used as shown in the table below.

Variogram	$\gamma_1(h)$	$\gamma_2(h)$	$\gamma_3(h)$
Range, a	4	9	16

The results of kriging using the three sets of data with different variogram models are given in the table below. The center block values are given in the first row. The estimated values in 8 small cells are listed and the block average of the small cells is calculated in the last row for each case. The block datum is exactly reproduced in all cases, which numerically confirms the theorem presented above. The changes in block data, well data and variogram models will affect estimates in small cells, but do not affect the exact reproduction of block datum.

	Set 1: Zcb = 0.25			Set 2: Zcb = 1.00			Set 3: Zcb = 4.00		
Cell	γ_1	γ_2	γ_3	γ_1	γ_2	γ_3	γ_1	γ_2	γ_3
1	0.18	0.15	0.14	0.95	0.93	0.93	3.99	4.06	4.11
2	0.28	0.26	0.25	1.01	0.98	0.95	3.96	3.83	3.75
3	0.17	0.14	0.13	0.94	0.93	0.93	3.99	4.09	4.15
4	0.26	0.24	0.23	1.00	0.97	0.95	3.97	3.88	3.82
5	0.22	0.24	0.25	0.98	1.01	1.04	4.02	4.12	4.18
6	0.35	0.39	0.40	1.08	1.10	1.09	4.02	3.92	3.86
7	0.21	0.22	0.23	0.97	1.00	1.03	4.02	4.14	4.22
8	0.32	0.36	0.38	1.06	1.08	1.08	4.02	3.96	3.91
Zavg	0.25	0.25	0.25	1.00	1.00	1.00	4.00	4.00	4.00

The next exercise is to check the spatial continuity changes after downscaling. A small 2D grid with a scale of 4m x 4m is downscaled to 1m x 1m using kriging. The large scale model and the small scale model are plotted in the Figure 1. The low and high value trends in the large scale model are preserved in the fine scale model and the values of small cells change more smoothly. All the block data are exactly reproduced.

A Large Example

A study area of about 4 sections (each section is 1 square mile): 3200m x 3200m. The porosity block data is taken from a large 2D prior model at the scale of 50m by 50m. The map of the block data is shown in Figure 2. There are 16 wells in this study area with small scale data. The locations of the 16 wells and the histogram of the well log data are shown in Figure 3. The horizontal and vertical variograms are calculated from the well log data, and the variogram model is shown in Figure 4.

The 2-D block data is extended to a 3-D model using the EDMK kriging method. The block data are downscaled in the vertical direction to generate a 3-D model at scale of 50m by 50m by 1m. The 3-D porosity model is shown in Figures 5 and 6. The original porosity map and all odd numbered x-y view slices of the 3-D model are shown in Figure 5. The dash lines in the original porosity map (the top left in Figure 5) indicate the locations of the x-z and y-z cross sections. The x-z and y-z cross sections of the 3-D Model are shown in Figure 6. The dashed lines in the x-z and y-z cross sections indicate the wells. The well data control the generation of the horizontal continuity in the cross sections.

The 3-D model is converted back to 2-D model using the arithmetic averaging of each column. The results are plotted together with the original 2-D porosity map in Figure 7. After downscaling to a 3-D model and upscaling back to a 2-D model, the new 2-D map is exactly same as the original map. The consistency between the two models indicates that the scaling methods are exact, and provide confidence in dealing with multiscale data.

Considerations for the downscale techniques

The large scale and fine scale data are assumed to be representative. The well data, measured as point data, are assumed to be representative to the fine scale volume. The multiscale data are assumed to be stationary so that the statistical properties of each scale are the same over the entire area and the statistical properties of different scales follow a same relationship over the entire area. If trends exist, subdividing the study area may help. The scale relationship of data is a linear. Upscaling of data and covariance gives a unique solution while downscaling has multiple solutions. The statistical properties at the small scale provide constraints on downscaling.

Reservoirs are often modeled at a large scale first to account for large scale trends and seismic data at a large scale. Then, reservoir models are built at smaller scales consistent with high resolution well data. The models at different scales will not be consistent with one another because different techniques and statistical parameters are used in model construction, that is, scaling the small scale models up to the scale of the large scale models does not match the large scale model results. This situation could raise doubts on both models and create a further problem of choosing a model. Exact scaling techniques ensures consistency between the models at different scales. An important assumption is that *the large scale models constructed directly at a large scale are more reliable than upscaled small scale models*.

In the exact downscaling technique, the direct sequential simulation approach is used. The "direct" means no Gaussian transformation of the data. The non-linear normal transform should be avoided when dealing with multiscale data or models with different scales. The reason is that after normal transform, the linear relationship of data at different scales is no long true.

$$Z_V = \frac{1}{n}\sum_{i=1}^{n} Z_i$$

$$\varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(Z_i)\right) \neq Z_V$$

Where φ(f) is a non-linear transform.

Conclusions

The large scale data can be exact downscaled so that the fine scale model can be consistent with the large scale model. The exact downscaling techniques are based on the kriging and the direct sequential simulation; multiscale data are used. The block data are exact reproduced and the heterogeneity at fine scale is captured by the point data and covariance model at small scale.

References

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Zanon, S., Nguyen, H. and Deutsch, C.V. 2002, power law averaging revisited, CCG report four, 2001/2002.

Tran, T.T., Deutsch, C.V. and Xie, Y., 2001, Direct Geostatistical Simulation with Multiscale Well, Seismic, and Production Data, SPE paper 71323.

Appendix A – Some Demonstrations

Analytical Kriging Results

We start with a block contains two constituent points to get Equation (4). Then, an additional point is considered with the two points block to show if the additional data affect the results. Finally, a block with three constituting points is used to get the equation (4).

One block contains two constituting points

The block datum is $Z_b = \frac{Z(\mathbf{u_1}) + Z(\mathbf{u_2})}{2}$.

using kriging with the block datum gives the estimates at locations $\mathbf{u_1}$ and $\mathbf{u_2}$:

$$\begin{aligned} Z^*(\mathbf{u_1}) - m &= \lambda_1 (Z_b - m) \\ Z^*(\mathbf{u_2}) - m &= \lambda_2 (Z_b - m) \end{aligned} \tag{8}$$

where the weights λ_1 and λ_2 from the kriging equation:

$$\begin{aligned} C_{bb}\lambda_1 &= C_{1b} \\ C_{bb}\lambda_2 &= C_{2b} \end{aligned} \tag{9}$$

For the two points block (n=2), from the equations (6) and (7):

$$C_{1b} = \frac{C_{11} + C_{12}}{2} = \frac{\sigma^2 + C_{12}}{2}$$

$$C_{2b} = \frac{C_{21} + C_{22}}{2} = \frac{\sigma^2 + C_{12}}{2}$$

$$C_{bb} = \frac{C_{11} + C_{12} + C_{21} + C_{22}}{4} = \frac{\sigma^2 + C_{12}}{2}$$

thus,

$$C_{bb} = C_{1b} = C_{2b} \tag{10}$$

insert (10) into equation (9), gives $\lambda_1 = \lambda_2 = 1$.

then, equation (8) becomes:

$$\begin{aligned} Z^*(\mathbf{u_1}) &= Z_b \\ Z^*(\mathbf{u_2}) &= Z_b \end{aligned}$$

therefore, the estimated block value:

$$Z_b^* = \frac{Z^*(\mathbf{u_1}) + Z^*(\mathbf{u_2})}{2} = \frac{Z_b + Z_b}{2} = Z_b$$

The estimated block value is exactly the block datum.

One block contains two constituting points plus an extra point

The block datum still is $Z_b = \frac{Z(\mathbf{u_1}) + Z(\mathbf{u_2})}{2}$.

using kriging with the block datum and the extra point gives the estimates at $\mathbf{u_1}$ and $\mathbf{u_2}$:

$$\begin{aligned} Z^*(\mathbf{u_1}) - m &= \lambda_1 (Z_b - m) + \mu_1 (Z(\mathbf{u_3}) - m) \\ Z^*(\mathbf{u_2}) - m &= \lambda_2 (Z_b - m) + \mu_2 (Z(\mathbf{u_3}) - m) \end{aligned} \tag{11}$$

the kriging equation systems for each location are:

$$\begin{aligned} C_{bb}\lambda_1 + C_{3b}\mu_1 &= C_{1b} \\ C_{3b}\lambda_1 + C_{33}\mu_1 &= C_{13} \end{aligned} \tag{12}$$

$$\begin{aligned} C_{bb}\lambda_2 + C_{3b}\mu_2 &= C_{2b} \\ C_{3b}\lambda_2 + C_{33}\mu_2 &= C_{23} \end{aligned} \tag{13}$$

add (12) to (13):

$$\begin{aligned} C_{bb}(\lambda_1 + \lambda_2) + C_{3b}(\mu_1 + \mu_2) &= C_{1b} + C_{2b} \\ C_{3b}(\lambda_1 + \lambda_2) + C_{33}(\mu_1 + \mu_2) &= C_{13} + C_{23} \end{aligned} \tag{14}$$

if let $(\mu_1 + \mu_2) = 0$, then (14) becomes:

$$\begin{aligned} \lambda_1 + \lambda_2 &= (C_{1b} + C_{2b}) / C_{bb} \\ \lambda_1 + \lambda_2 &= (C_{13} + C_{23}) / C_{3b} \end{aligned}$$

recall equation (10): $C_{bb} = C_{1b} = C_{2b}$.

And from the equation (7), n=2:

$$C_{3b} = \frac{C_{31} + C_{32}}{2} = \frac{C_{13} + C_{23}}{2}$$

thus, $(\lambda_1 + \lambda_2) = 2$.

And $\begin{aligned} \lambda_1 + \lambda_2 &= 2 \\ \mu_1 + \mu_2 &= 0 \end{aligned}$ is the solution of the equations (14).

Since the kriging system is unique and only has a unique solution, the founded solution must be the correct unique solution.

then, the estimated block value:

$$
\begin{aligned}
Z_b^* &= \frac{Z^*(\mathbf{u_1}) + Z^*(\mathbf{u_2})}{2} \\
&= \frac{\lambda_1(Z_b - m) + \mu_1(Z(\mathbf{u_3}) - m) + m + \lambda_2(Z_b - m) + \mu_2(Z(\mathbf{u_3}) - m) + m}{2} \\
&= \frac{(\lambda_1 + \lambda_2)(Z_b - m) + (\mu_1 + \mu_2)(Z(\mathbf{u_3}) - m) + 2m}{2} = \frac{2(Z_b - m) + 2m}{2} \\
&= Z_b
\end{aligned}
$$

The block datum is exactly reproduced and the extra datum does not affect the estimated results.

One block contains three constituent points

The block datum:

$$Z_b = \frac{Z(\mathbf{u_1}) + Z(\mathbf{u_2}) + Z(\mathbf{u_3})}{3}$$

using kriging with the block data gives the estimates at $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$:

$$
\begin{aligned}
Z^*(\mathbf{u_1}) - m &= \lambda_1(Z_b - m) \\
Z^*(\mathbf{u_2}) - m &= \lambda_2(Z_b - m) \\
Z^*(\mathbf{u_3}) - m &= \lambda_3(Z_b - m)
\end{aligned}
\tag{15}
$$

where the weights λ_1, λ_2 and λ_3 from the kriging equation:

$$
\begin{aligned}
C_{bb}\lambda_1 &= C_{1b} \\
C_{bb}\lambda_2 &= C_{2b} \\
C_{bb}\lambda_3 &= C_{3b}
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
\lambda_1 &= C_{1b} / C_{bb} \\
\lambda_2 &= C_{2b} / C_{bb} \\
\lambda_3 &= C_{3b} / C_{bb}
\end{aligned}
\tag{16}
$$

For the three points block (n = 3), from the equations (6) and (7):

$$
\begin{aligned}
C_{1b} &= \frac{C_{11} + C_{12} + C_{13}}{3} = \frac{\sigma^2 + C_{12} + C_{13}}{3} \\
C_{2b} &= \frac{C_{21} + C_{22} + C_{23}}{3} = \frac{C_{21} + \sigma^2 + C_{23}}{3} \\
C_{3b} &= \frac{C_{31} + C_{32} + C_{33}}{3} = \frac{C_{31} + C_{32} + \sigma^2}{3} \\
C_{bb} &= \frac{C_{11} + C_{12} + C_{13} + C_{21} + C_{22} + C_{23} + C_{31} + C_{32} + C_{33}}{9} = \frac{3\sigma^2 + 2C_{12} + 2C_{13} + 2C_{23}}{9}
\end{aligned}
\tag{17}
$$

From equations (16) and (17), we can get:

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{C_{1b}}{C_{bb}} + \frac{C_{2b}}{C_{bb}} + \frac{C_{3b}}{C_{bb}}$$
$$= \frac{(C_{11} + C_{12} + C_{13})/3 + (C_{21} + C_{22} + C_{23})/3 + (C_{31} + C_{32} + C_{33})/3}{(3\sigma^2 + 2C_{12} + 2C_{13} + 2C_{23})/9}$$
$$= \frac{(3\sigma^2 + 2C_{12} + 2C_{13} + 2C_{23})/3}{(3\sigma^2 + 2C_{12} + 2C_{13} + 2C_{23})/9}$$
$$= 3$$

then, the estimated block value:

$$Z_b^* = \frac{Z^*(\mathbf{u_1}) + Z^*(\mathbf{u_2}) + Z^*(\mathbf{u_3})}{3} = \frac{\lambda_1(Z_b - m) + m + \lambda_2(Z_b - m) + m + \lambda_3(Z_b - m) + m}{3}$$
$$= \frac{(\lambda_1 + \lambda_2 + \lambda_3)(Z_b - m) + 3m}{3} = \frac{3(Z_b - m) + 3m}{3} = Z_b$$

The block data is reproduced exactly regardless of C(**h**).

We have show that the block data can be reproduced exactly with kriging of block data for n=2, n=2 with an extra point, and n=3. Therefore, it can be inferred that estimating all locations constituting the block by kriging can always reproduce block data. i.e. when $Z^*(\mathbf{u_i})$ values are based on Z_b,

$$\frac{1}{n}\sum_{i=1}^{n} Z^*(\mathbf{u_i}) = Z_b$$

Analytical Sequential Simulation Results

Firstly, we start with a block contains two constituting points to get the equation (4). Then, an additional point is considered with the two points block to show if the additional data affect the results. Thirdly, a block with three constituting points is used to get the equation (4).

One block contains two constituent points

The block value:

$$Z_b = \frac{Z(\mathbf{u_1}) + Z(\mathbf{u_2})}{2}$$

To simplify the equations, assume the mean is zero.

At the first location $\mathbf{u_1}$:

the simple kriging estimate and equation are

$$Z^*(\mathbf{u_1}) = \lambda_1 Z_b$$

$$C_{b\,b}\lambda_1 = C_{1b}$$

recall equation (10), $C_{1b} = C_{bb}$, so $\lambda_1 = \dfrac{C_{1b}}{C_{b\,b}} = 1$

The simulated value: $Z^s(\mathbf{u_1}) = \lambda_1 Z_b + R(\mathbf{u_1}) = Z_b + R(\mathbf{u_1})$

where $R(\mathbf{u_1})$ is a random residual drawn by Mont Coral Simulation. $R(\mathbf{u_1})$ follows a normal distribution with mean of zero and variance of $\sigma_k^2(\mathbf{u_1})$.

At next location $\mathbf{u_2}$:

the kriging estimate and equations:

$$Z^*(\mathbf{u_2}) = \lambda_2 Z_b + w\, Z^s(\mathbf{u_1}) = \lambda_2 Z_b + w\, Z_b + w\, R(\mathbf{u_1})$$

$$\begin{aligned} \lambda_2 C_{b\,b} + w\, C_{1b} &= C_{2b} \\ \lambda_2 C_{1b} + w\, C_{1\,1} &= C_{2\,1} \end{aligned} \qquad (18)$$

recall equation (10): $C_{b\,b} = C_{1b} = C_{2b} = \dfrac{\sigma^2 + C_{1\,2}}{2}$.

Then, equation (18) becomes:

$\lambda_2 + w = 1$

$\lambda_2 \left(\dfrac{\sigma^2 + C_{1\,2}}{2}\right) + w\, C_{1\,1} = C_{2\,1}$

We can get

$\lambda_2(\sigma^2 + C_{1\,2}) + 2\sigma^2(1 - \lambda_2) = 2C_{2\,1}$

the equation can be solved to get $\lambda_2 = 2$ and $w = -1$.

and the simple kriging variance is:

$$\sigma_k^2(\mathbf{u_2}) = C_{2\,2} - \lambda_2 C_{2b} - w\, C_{2\,1} = \sigma^2 - 2\,\frac{(C_{1\,2} + \sigma^2)}{2} + C_{1\,2} = 0$$

then the random residual $R(\mathbf{u}_2) \equiv 0$ because it follows a normal distribution with mean of zero and variance of $\sigma_k^2(\mathbf{u_2}) = 0$.

The simulated value: $Z^s(\mathbf{u_2}) = Z^*(\mathbf{u_2}) = \lambda_2 Z_b + w\, Z^s(\mathbf{u_1}) = 2Z_b - Z^s(\mathbf{u_1})$

Then, the simulated block value is:

$$Z_b^s = \frac{Z^s(\mathbf{u_1}) + Z^s(\mathbf{u_2})}{2} = \frac{Z^s(\mathbf{u_1}) + 2Z_b - Z^s(\mathbf{u_1})}{2} = Z_b$$

The block datum is reproduced exactly for a block contains two points.

One block contains two constituent points plus an extra point

The block value:

$$Z_b = \frac{Z(\mathbf{u_1}) + Z(\mathbf{u_2})}{2}$$

To simplify the equations, assume the mean is zero.

At the first location $\mathbf{u_1}$:

The simple kriging estimate using the block data and the extra point:

$$Z^*(\mathbf{u_1}) = \lambda_1 Z_b + \mu_1 Z(\mathbf{u_3})$$

The simulated value: $Z^s(\mathbf{u_1}) = \lambda_1 Z_b + \mu_1 Z(\mathbf{u_3}) + R(\mathbf{u_1})$

where $R(\mathbf{u_1})$ is a random residual drawn by Mont Coral Simulation. $R(\mathbf{u_1})$ follows a normal distribution with mean of zero and variance of $\sigma_k^2(\mathbf{u_1})$.

At next location $\mathbf{u_2}$:

The simple kriging estimate:

$$Z^*(\mathbf{u_2}) = \lambda_2 Z_b + \mu_2 Z(\mathbf{u_3}) + w_1 Z^s(\mathbf{u_1})$$

And the system of equations:

$$\begin{aligned} &\lambda_2 C_{bb} + \mu_2 C_{1b} + w_1 C_{3b} = C_{2b} \ldots\ldots\ldots (a) \\ &\lambda_2 C_{1b} + \mu_2 C_{11} + w_1 C_{13} = C_{21} \ldots\ldots\ldots (b) \\ &\lambda_2 C_{3b} + \mu_2 C_{31} + w_1 C_{33} = C_{23} \ldots\ldots\ldots (c) \end{aligned} \qquad (19)$$

from (19a):

$$\lambda_2 = \frac{C_{bb} - C_{bb}\mu_2 - C_{3b} w_1}{C_{bb}} = 1 - \mu_2 - \frac{C_{3b}}{C_{bb}} w_1 \qquad (20)$$

(19a) - (19b):

$$(C_{bb} - \sigma^2)\mu_2 + (C_{3b} - C_{13})w_1 = C_{bb} - C_{21} \tag{21}$$

With equation (10): $C_{bb} = C_{1b} = C_{2b} = \frac{\sigma^2 + C_{12}}{2}$ and $C_{3b} = \frac{C_{31} + C_{32}}{2}$, we can get:

$$\mu_2 = \frac{(C_{bb} - C_{21}) - (C_{3b} - C_{13})w_1}{C_{bb} - \sigma^2} = \frac{(\sigma^2 - C_{12}) - (C_{32} - C_{31})w_1}{C_{bb} - \sigma^2}$$
$$= \frac{(C_{31} - C_{32})}{C_{12} - \sigma^2} w_1 - 1 \tag{22}$$

$$w_1 = \frac{(C_{bb} - C_{21}) - (C_{bb} - \sigma^2)\mu_2}{C_{3b} - C_{13}} = \frac{(\sigma^2 - C_{12}) - (\sigma^2 - C_{12})\mu_2}{C_{32} - C_{13}}$$
$$= \frac{(\sigma^2 - C_{12})(1 + \mu_2)}{C_{32} - C_{13}} \tag{23}$$

Insert (20) into (19c):

$$C_{3b}(1 - \mu_2 - \frac{C_{3b}}{C_{bb}} w_1) + C_{13}\mu_2 + \sigma^2 w_1 = C_{23}$$

rearrange it:

$$(C_{31} - C_{32}) + (C_{31} - C_{32})\mu_2 + (\sigma^2 - \frac{C_{3b}^2}{C_{bb}})w_1 = 0 \tag{24}$$

Insert (22) into (24):

$$(C_{31} - C_{32}) + (C_{31} - C_{32})(\frac{(C_{31} - C_{32})}{C_{12} - \sigma^2} w_1 - 1) + (\sigma^2 - \frac{C_{3b}^2}{C_{bb}})w_1 = 0$$

rearrange it:

$$\frac{(C_{31} - C_{32})^2}{C_{12} - \sigma^2} w_1 + (\frac{\sigma^2 C_{bb} - C_{3b}^2}{C_{bb}})w_1 = 0$$

this equation gives $w_1 = 0$.

Then, from (22), $\mu_2 = -1$.

And from (20), $\lambda_2 = 2$.

$$\lambda_2 = 2$$

Therefore, the solution of the kriging system is: $\mu_2 = -1$

$$w_1 = 0$$

the simple kriging variance is:

$$\begin{aligned}\sigma_k^2(\mathbf{u_2}) &= C_{22} - \lambda_2 C_{2b} - \mu_2 C_{21} - w_1 C_{23} \\ &= \sigma^2 - 2C_{2b} + C_{21} = \sigma^2 - (\sigma^2 + C_{21}) + C_{21} \\ &= 0\end{aligned}$$

then the random residual $R(\mathbf{u}_2) \equiv 0$ because it follows a normal distribution with mean of zero and variance of $\sigma_k^2(\mathbf{u_2}) = 0$.

The simulated value:

$$Z^s(\mathbf{u_2}) = Z^*(\mathbf{u_2}) + R(\mathbf{u_2}) = \lambda_2 Z_b + \mu_2 Z^s(\mathbf{u_1}) + w_1 Z(\mathbf{u_3}) = 2Z_b - Z^s(\mathbf{u_1})$$

Therefore, the simulated block value is

$$Z_b^s = \frac{Z^s(\mathbf{u_1}) + Z^s(\mathbf{u_2})}{2} = \frac{Z^s(\mathbf{u_1}) + 2Z_b - Z^s(\mathbf{u_1})}{2} = Z_b$$

The block datum is exactly reproduced and the extra datum does not affect the estimated results.

One block contains three constituent points

The block value:

$$Z_b = \frac{Z(\mathbf{u_1}) + Z(\mathbf{u_2}) + Z(\mathbf{u_3})}{3}$$

To simplify the equations, assume the mean is zero.

At the first location $\mathbf{u_1}$:

The simple kriging estimate:

$$Z^*(\mathbf{u_1}) = \lambda_1 Z_b$$

The simulated value: $Z^s(\mathbf{u_1}) = \lambda_1 Z_b + R(\mathbf{u_1})$

where $R(\mathbf{u_1})$ is a random residual drawn by Mont Coral Simulation. $R(\mathbf{u_1})$ follows a normal distribution with mean of zero and variance of $\sigma_k^2(\mathbf{u_1})$.

At next location $\mathbf{u_2}$:

the kriging estimate and equations:

$$Z^*(\mathbf{u}_2) = \lambda_2 Z_b + w_1 Z_s(\mathbf{u}_1) = \lambda_2 Z_b + w_1 \lambda_1 Z_b + w_1 R(\mathbf{u}_1)$$

The simulated value: $Z^s(\mathbf{u}_2) = \lambda_2 Z_b + w_1 Z^s(\mathbf{u}_1) + R(\mathbf{u}_2)$

where $R(\mathbf{u}_2)$ is a random residual drawn by Mont Coral Simulation. $R(\mathbf{u}_2)$ follows a normal distribution with mean of zero and variance of $\sigma_k^2(\mathbf{u}_2)$.

At the last location $\mathbf{u}_3$:

the kriging estimate:

$$Z^*(\mathbf{u}_3) = \lambda_3 Z_b + w_2 Z^s(\mathbf{u}_1) + w_3 Z^s(\mathbf{u}_2)$$

and the kriging system:

$$\begin{aligned} &\lambda_3 C_{bb} + w_2 C_{1b} + w_3 C_{2b} = C_{3b} \ldots\ldots (a) \\ &\lambda_3 C_{1b} + w_2 C_{11} + w_3 C_{12} = C_{13} \ldots\ldots (b) \\ &\lambda_3 C_{2b} + w_2 C_{21} + w_3 C_{22} = C_{23} \ldots\ldots (c) \end{aligned} \qquad (25)$$

With equations (17), (25) becomes:

$$\begin{aligned} &\lambda_3(3\sigma^2 + 2C_{12} + 2C_{13} + 2C_{23}) + w_2 3(\sigma^2 + C_{12} + C_{13}) \\ &+ w_3 3(\sigma^2 + C_{21} + C_{23}) = 3(\sigma^2 + C_{31} + C_{32}) \ldots\ldots\ldots\ldots\ldots (a) \\ &\lambda_3(\sigma^2 + C_{12} + C_{13}) + w_2 3\sigma^2 + w_3 3C_{12} = 3C_{31} \ldots\ldots\ldots\ldots\ldots (b) \\ &\lambda_3(\sigma^2 + C_{21} + C_{23}) + w_2 3C_{12} + w_3 3\sigma^2 = 3C_{32} \ldots\ldots\ldots\ldots\ldots (c) \end{aligned} \qquad (26)$$

(26b) - (26c):

$$\lambda_3(C_{13} - C_{23}) + w_2(3\sigma^2 - 3C_{12}) + w_3(3C_{12} - 3\sigma^2) = 3C_{31} - 3C_{32}$$

rearrange it:

$$(\lambda_3 - 3)(C_{13} - C_{23}) + (w_2 - w_3)(3\sigma^2 - 3C_{12}) = 0$$

from the equations we can get $\lambda_3 = 3$ and $w_2 = w_3$.

Then from (26b) or (26c), we can get $w_2 = w_3 = -1$.

$$\lambda_3 = 2$$

Therefore, the solution of the kriging system is: $w_2 = -1$

$$w_3 = -1$$

and the simple kriging variance is:

$$\sigma_k^2(\mathbf{u}_3) = C_{33} - \lambda_3 C_{3b} - w_2 C_{31} - w_3 C_{32} = \sigma^2 - (\sigma^2 + C_{31} + C_{32}) + C_{31} + C_{32} = 0$$

then the random residual $R(\mathbf{u}_3) \equiv 0$ because it follows a normal distribution with mean of zero and variance of $\sigma_k^2(\mathbf{u}_3) = 0$.

The simulated value:

$$Z^s(\mathbf{u}_3) = Z^*(\mathbf{u}_3) = \lambda_3 Z_b + w_2 Z^s(\mathbf{u}_1) + w_3 Z^s(\mathbf{u}_2) = 3Z_b - Z^s(\mathbf{u}_1) - Z^s(\mathbf{u}_2)$$

Therefore, the simulated block value is

$$Z_b^s = \frac{Z^s(\mathbf{u_1}) + Z^s(\mathbf{u_2}) + Z^s(\mathbf{u_3})}{3} = \frac{Z^s(\mathbf{u_1}) + Z^s(\mathbf{u_2}) + 3Z_b - Z^s(\mathbf{u_1}) - Z^s(\mathbf{u_2})}{3} = Z_b$$

The block data is reproduced exactly with simulation of block data when n=3.

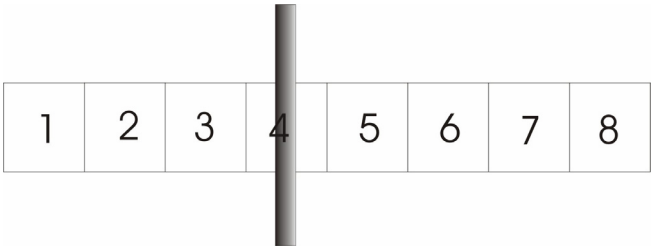
We have shown that the block data can be reproduced exactly with simulation of block data for n=2, n=2 with an extra point and n=3. Therefore, it can be inferred that estimating all locations constituting the block by sequential simulation can always reproduce block data. i.e. when $Z^s(\mathbf{u_i})$ values are based on Z_b,

$$Z_b^s = \frac{1}{n}\sum_{i=1}^{n} Z^s(\mathbf{u_i}) = Z_b$$

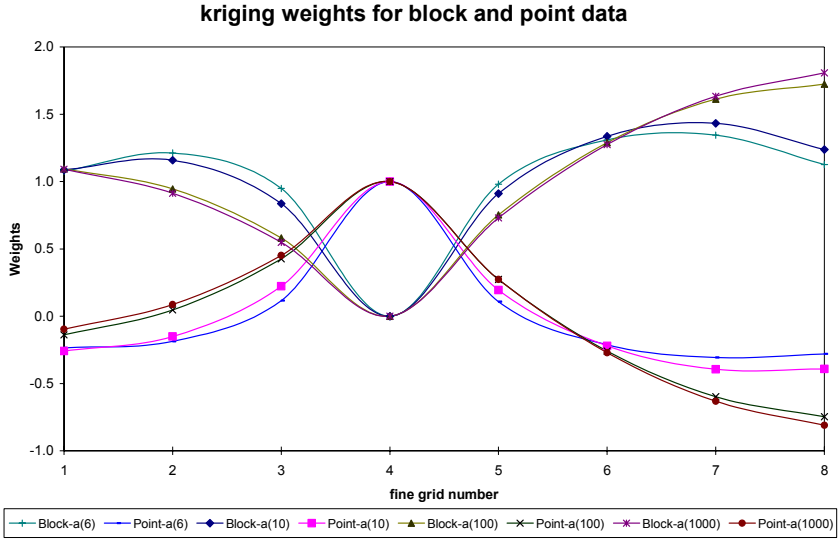
Appendix B – Some Interesting Numerical Results

Single Block Case

A block is discretized to 8 finer cells. A well is located at the center of cell 4. Different variograms can affect the kriging weights. Three variograms ranges are used for the example: (1) small range: ah=av=10, (2) middle range: ah=av=100, (3) long range: ah=av=1000. The results of using the block kriging approach (weights of block and the well data) are listed in the table and plot in the following figure:



cell no.	ah = av = 6		ah = av = 10		ah = av = 100		ah = av = 1000	
	block	point	block	point	block	point	block	point
1	1.08	-0.24	1.09	-0.26	1.10	-0.14	1.09	-0.10
2	1.21	-0.19	1.16	-0.15	0.95	0.05	0.91	0.09
3	0.95	0.12	0.84	0.22	0.58	0.43	0.55	0.45
4	0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00
5	0.98	0.11	0.91	0.19	0.75	0.27	0.73	0.27
6	1.31	-0.21	1.34	-0.22	1.29	-0.26	1.27	-0.27
7	1.35	-0.31	1.43	-0.39	1.61	-0.60	1.63	-0.63
8	1.13	-0.28	1.24	-0.39	1.72	-0.75	1.81	-0.81



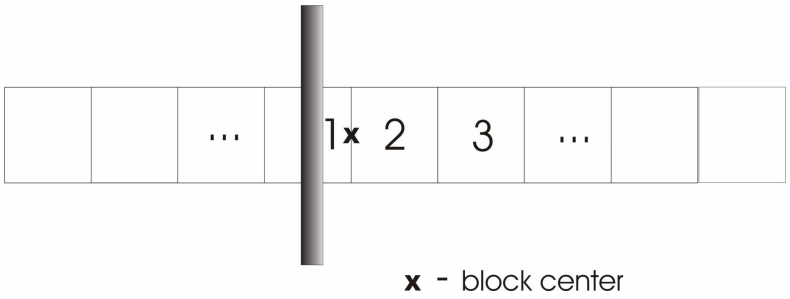
The weight of block datum is always larger than the weight of point data except at the well location. Far away from the well, the weight of the block is much larger than the weight of point

datum. The well gets all the weight in block 4, as expected. As the variogram range increases, the weight of the well datum increases and the weight of the block decreases.

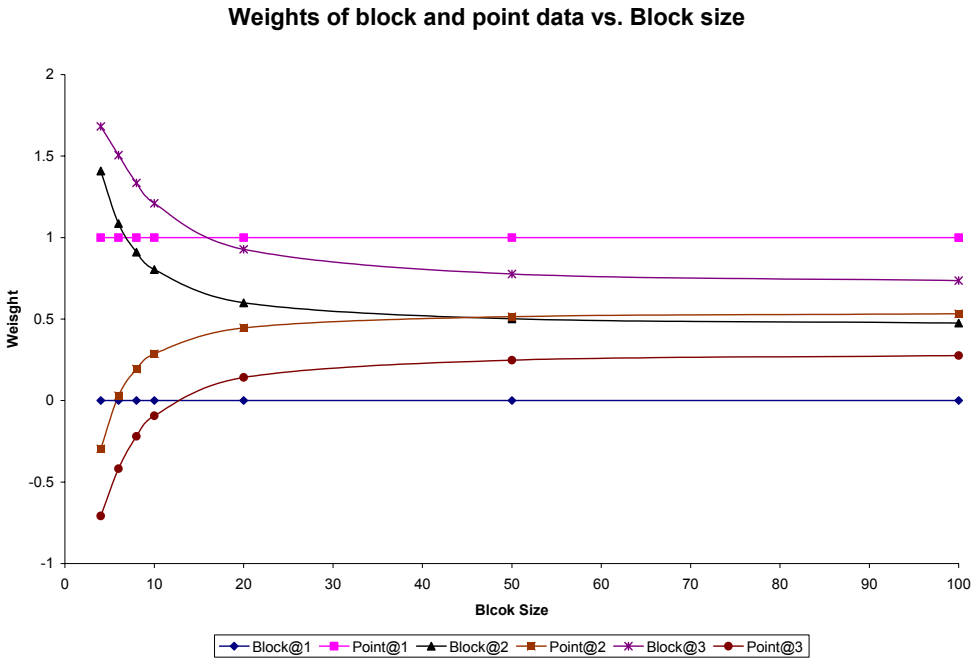
If the well is not at the center of the cell 4, the weights of block and point are different from the weights when the well is at center.

Block Size

The weights change with the block size; the weights in the three cells near the well are shown.

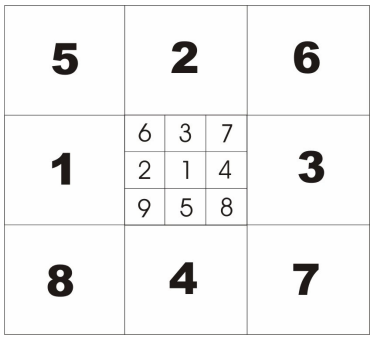


block size	4	6	8	10	20	50	100
block weight@1	0	0	0	0	0	0	0
point weight@1	1	1	1	1	1	1	1
block weight@2	1.41	1.09	0.91	0.80	0.60	0.50	0.47
point weight@2	-0.30	0.03	0.19	0.29	0.45	0.51	0.53
block weight@3	1.68	1.51	1.34	1.21	0.93	0.78	0.74
point weight@3	-0.71	-0.42	-0.22	-0.09	0.14	0.25	0.28

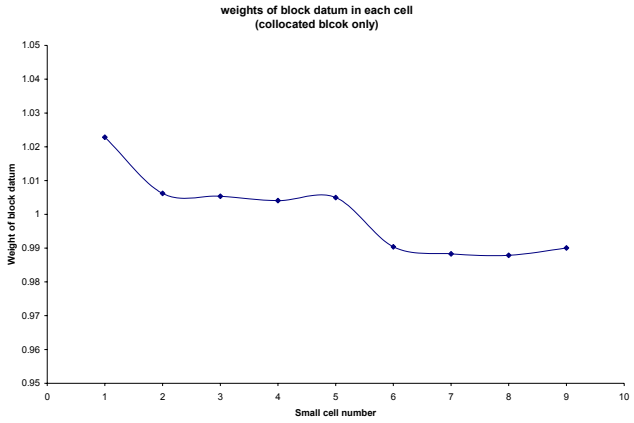


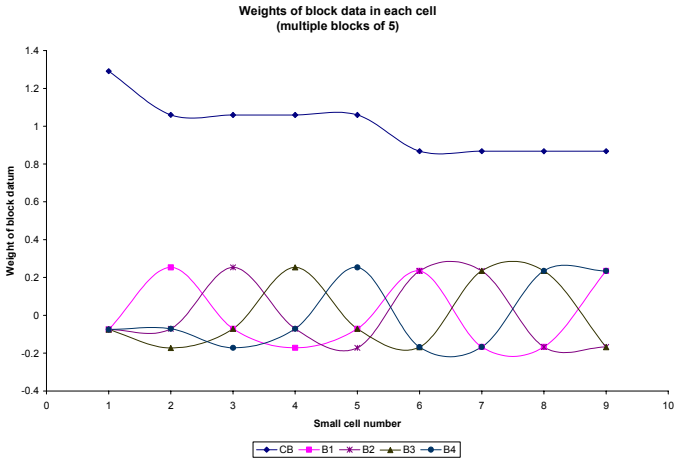
Screen Effect with Multiple Block Data

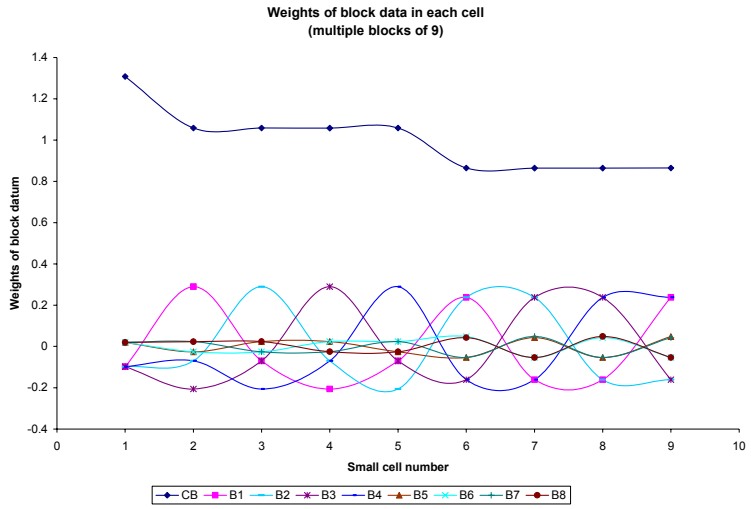
The center block is discretized to 9 small cells. And the surrounding large blocks are shown in the following figure. 1-4 blocks are directly connected, and 5-8 blocks are corner connected.



Kriging was performed for three cases: (1) the collocated block only; (2) the collocated block plus four neighboring blocks, and (3) all 9 surrounding blocks. The weights are shown below:

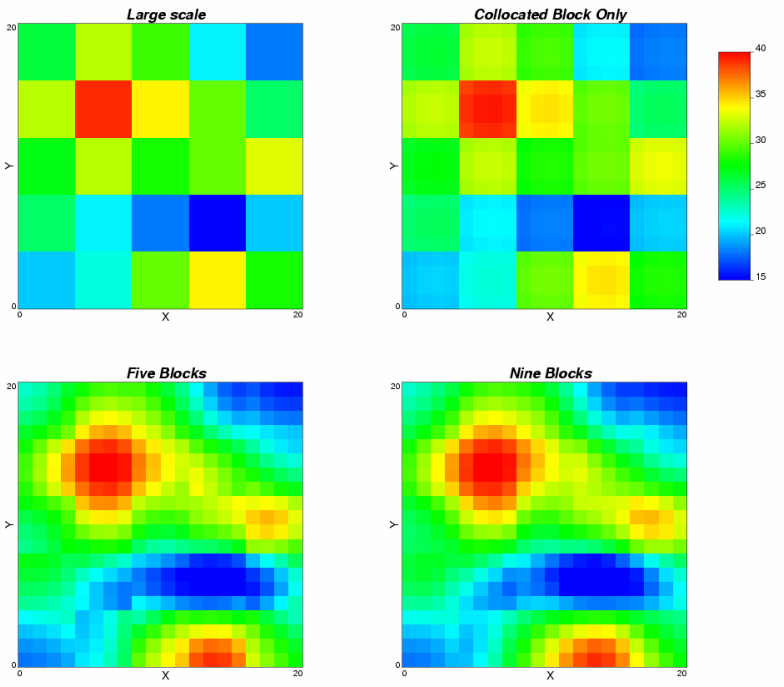






Kriging with Multiple Block Data

The figure below shows a 2-D grid of large scale data (upper left), the results of kriging small scale blocks with the collocated block only (upper right), the results of kriging with five and nine blocks. Clearly, using the collocated block alone does not provide any information on the spatial arrangement of small scale estimates.



There are a number of numerical problems (precision of covariance calculations and numerical storage) that remain to be overcome. At times, exact reproduction is compromised because of singular matrices and these numerical problems.

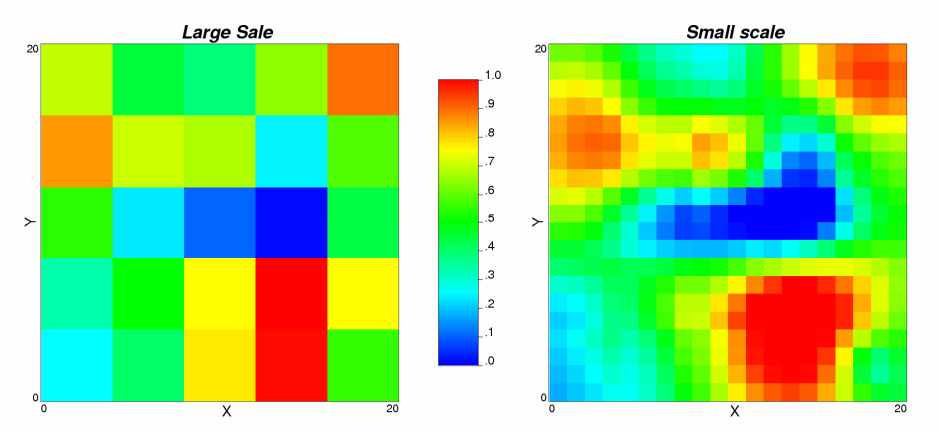


Figure 1: The model of large scale 4m x 4m (left) is downscaled to small scale 1m x 1m (right).

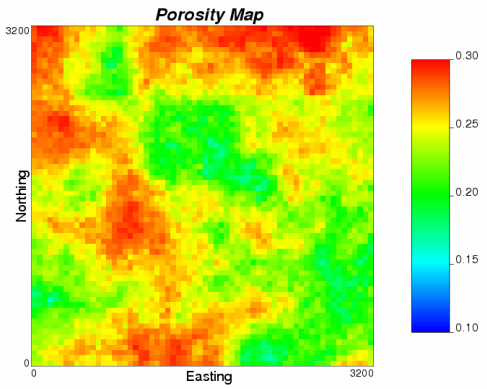


Figure 2: 2D map of porosity block data at scale of 50m x 50m x 10 m.

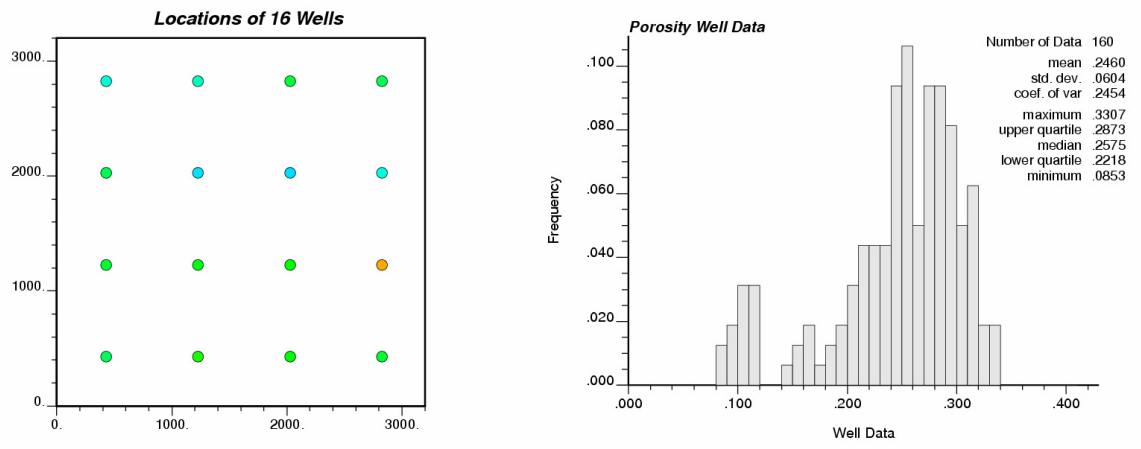


Figure 3: The location map of the 16 wells and the histogram of the well log data.

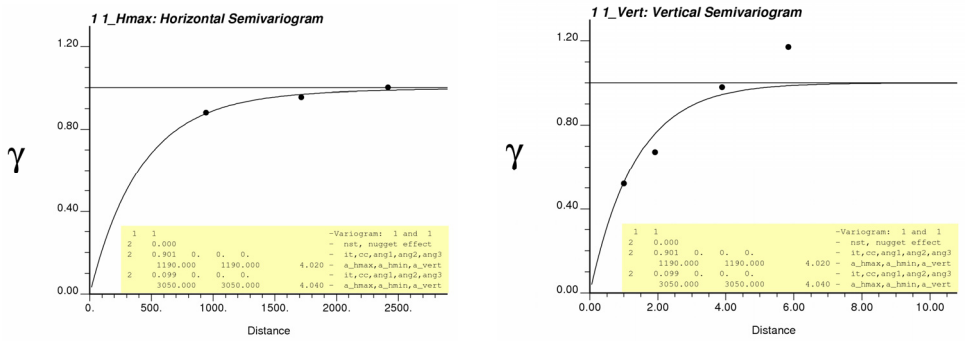


Figure 4: The horizontal and vertical variograms and variogram models.

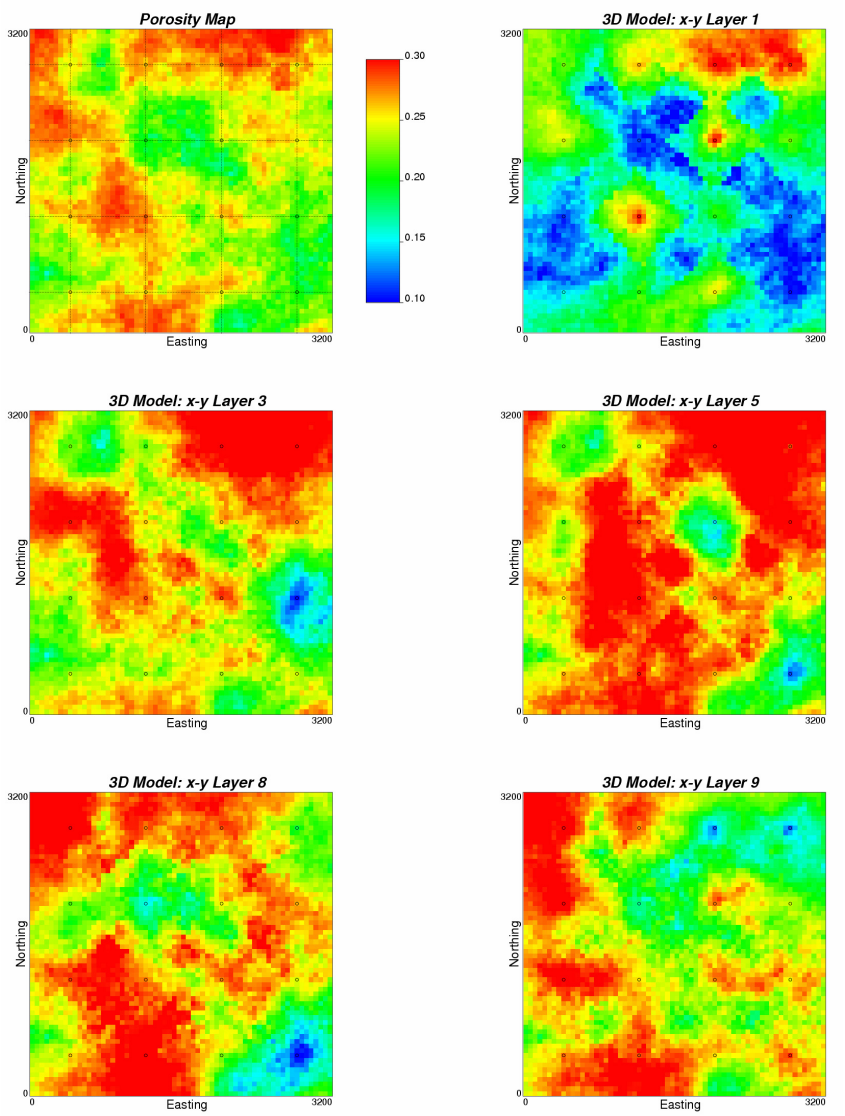


Figure 5: The 2-D porosity map and all odd number x-y view slices of the 3-D model. The 3D porosity is at scale of 50x50x1m. The dash lines in the porosity map (the top left) indicate the locations of the x-z and y-z cross sections.

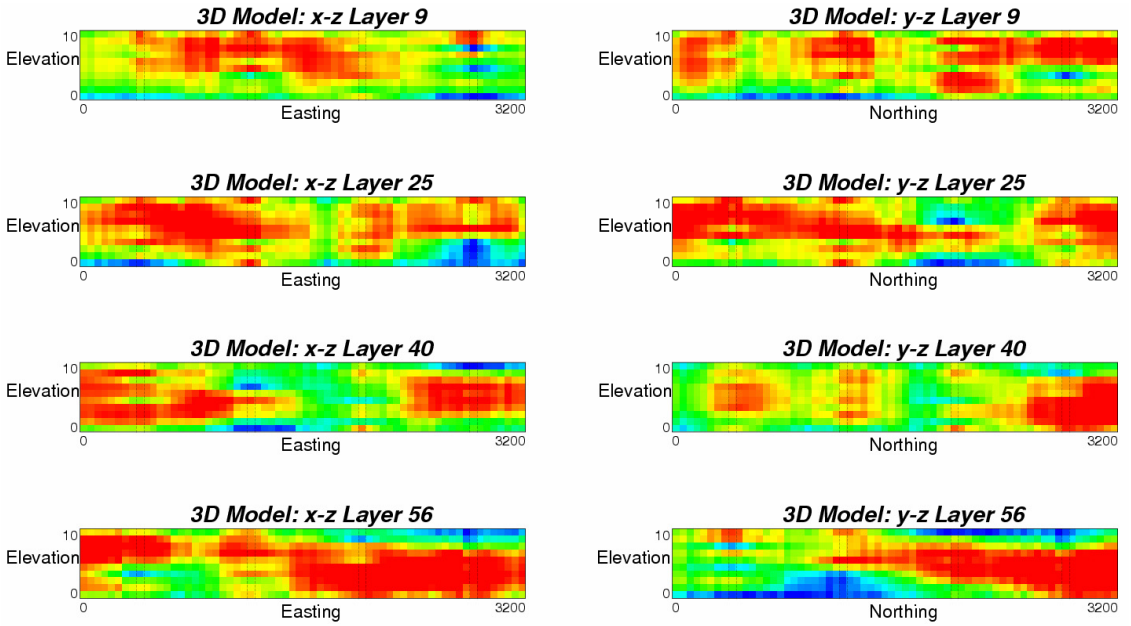


Figure 6: The x-z and y-z cross sections of the 3D porosity model. The 3D porosity is at scale of 50x50x1m. The dish lines in the x-z and y-z cross sections indicate the wells. The horizontal continuity is shown based on the well data.

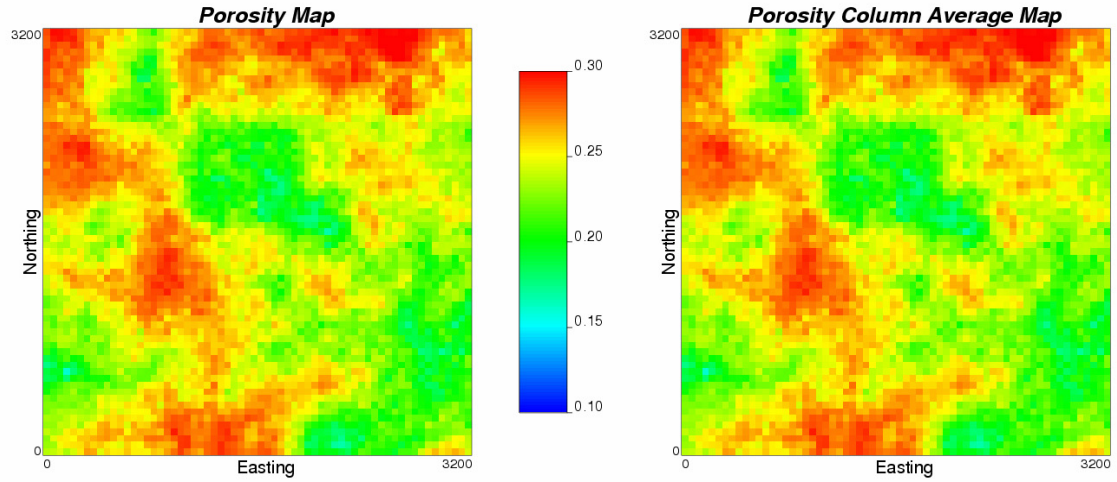


Figure 7: The original 2D porosity map (left) and the map of the average porosity of each column in the 3D model (right).